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THE EFFECTIVENESS OF USING ONE VARIANT OF THE FINITE-ELEMENT METHOD FOR SOLVING NONLINEAR NONSTATIONARY HEAT-CONDUCTION PROBLEMS

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It is established that the use of a previously proposed variant of the finite-element method for solving nonlinear nonstationary heat-conduction problems is efficient (in the sense of computer time expenditures) when the half-width of the band β of the resulting matrix in the system of linear algebraic equations does not exceed a certain value of β_{cr}. As a result of a numerical experiment for an axially symmetric case we obtain $\beta_{cr} \approx 47$, which demonstrates the effectiveness of this variant of the finite-element method for solving a wide range of practical problems of heat-conduction theory.

It is a known fact [1] that in numerical solution of nonlinear nonstationary heat-conduction problems by the finite-element method most of the computer time τ is spent in the formation of a system of linear algebraic equations τ_{form} and its solution τ_{sol} (i.e., $\tau = \tau_{form} + \tau_{sol}$). To reduce the computer time, various procedures are used: finite elements with a higher order of approximation [2]; schemes with a smaller number of integration points over the element space [3]; and effective methods for solving the resulting system of linear algebraic equations [1, 4, 5].

Below we will show that with certain restrictions on the width of the band of the matrix in the system of linear algebraic equations when using the variant of the finite-element method proposed in [6], it is also possible to reduce computer time as compared with the traditional scheme of the finite-element method. First we will present the relations that are required to compare computer time expenditures for the traditional finite-element method and the indicated variant of the finite-element method.

The solution of the heat-conduction problem

$$c(T)\rho(T)\frac{\partial T}{\partial t} = \operatorname{div}(\lambda(T)\operatorname{grad} T) + f(T, \mathrm{r}, t), \quad \mathrm{r} \in V, \quad t > 0;$$

$$T(\mathrm{r}, 0) = T_{in}(\mathrm{r}), \quad \mathrm{r} \in V; \quad T(\mathrm{r}, t) = \varphi(\mathrm{r}, t), \quad \mathrm{r} \in \Sigma_1, \quad t > 0;$$

$$\lambda\frac{\partial T}{\partial n}(\mathrm{r}, t) = -q(T, \mathrm{r}, t), \quad \mathrm{r} \in \Sigma_2, \quad t > 0; \tag{1}$$

$$\lambda\frac{\partial T}{\partial n}(\mathrm{r}, t) = \alpha(T, \mathrm{r}, t)(T_{med}(\mathrm{r}, t) - T(\mathrm{r}, t)), \quad \mathrm{r} \in \Sigma_3, \quad t > 0,$$

when using the traditional finite-element method in combination with the Galerkin method and approximation of the temperature T within the limits of each time interval $[t_n, t_{n+1}]$ $(n = 0, 1, 2, \ldots)$:

$$T = \omega T^{(n+1)} + (1 - \omega) T^{(n)} \quad (0 \le \omega \le 1) \tag{2}$$

is reduced to a solution, in each time layer, of a system of nonlinear algebraic equations

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$$[A]\{T\}^{(n+1)} = \{R\} \quad (n = 0, 1, 2, \ldots) \tag{3}$$

for the vector of the nodal values of the temperature $\{T\}^{(n+1)}$ at time $t = t_{n+1}$. In this case, the matrix $[A]$ and the vector on the right-hand side $\{R\}$ of system (3) are determined by the formulas

$$[A] = [C]/\Delta t_{n+1} + \omega([K] + [H]),$$

$$\{R\} = ([C]/\Delta t_{n+1} - (1 - \omega)([K] + [H]))\{T\}^{(n)} + \{f\} + \{\alpha\} + \{q\},$$

where $\Delta t_{n+1} = t_{n+1} - t_n$ $(n = 0, 1, 2, \ldots)$ is the time step. The elements of the matrices of heat capacity $[C]$, thermal conductivity $[K]$, and allowance for the third-kind boundary conditions $[H]$ and vectors $\{\alpha\}$, $\{q\}$, and $\{f\}$ are calculated by ensembling the magnitudes of the volumetric integrals

$$\int_{V^{(e)}} c(T)\rho(T) N_s N_i \, dV, \quad \int_{V^{(e)}} \lambda(T) \operatorname{grad} N_s \operatorname{grad} N_i \, dV, \tag{4}$$

$$e = 1, 2, \ldots, M, \quad s, i = 1, 2, \ldots, K^{(e)},$$

as well as of the surface and volumetric integrals

$$\int_{\Sigma_3^{(e)}} \alpha N_s N_i d\Sigma, \quad \int_{\Sigma_3^{(e)}} \alpha T_{\text{med}} N_s d\Sigma, \quad \int_{\Sigma_2^{(e)}} q N_s d\Sigma, \quad \int_{V^{(e)}} f N_s \, dV. \tag{5}$$

Here M is the number of all finite elements of the partitioning of region V; $K^{(e)}$ is the number of nodes of the e-th element of $V^{(e)}$.

Solution of nonlinear system (3) is usually reduced to determination of a certain (perhaps iterative) sequence of systems of linear algebraic equations

$$[A]_i \{T\}_{i+1}^{(n+1)} = \{R\}_i \quad (i = 1, 2, \ldots), \tag{6}$$

where $\{A\}_i$ and $\{R\}_i$ are the matrix and the right-hand side that are calculated for the temperature in the i-th iteration, $\{T\}_{i+1}^{(n+1)}$ is the temperature calculated in the next, $i + 1$-th iteration. The matrices $[A]_i$ have a band structure; they are symmetric and positive definite [7]. Effective methods (those of Kholetskii and Kraut [4, 5]) are used for solving such systems. However, in each iteration, except for integrals (5), it is necessary to find numerically integrals (4), and this is involves considerable computer time expenditures.

In contrast to the above-described traditional finite-element method, in the variant of the finite-element method suggested in [6] the corresponding volumetric integrals are calculated only once, at the beginning of calculations. According to [6], the solution of heat conduction problem (1) by introducing Goodman and Kirchhoff substitutions [8]

$$C = \int_{T_0}^{T} c(x)\rho(x) \, dx, \quad \Lambda = \int_{T_0}^{T} \lambda(x) \, dx \tag{7}$$

and using the finite-element method in combination with the Galerkin method is reduced to solution of the system of equations

$$[P]\{\dot{C}\} + [B]\{\Lambda\} + [H]\{T\} = \{S\} \tag{8}$$

with corresponding initial conditions. Here T_0 is a given constant with the dimensions of temperature; $[P]$ and $[B]$ are matrices of the N-th order (N is the number of all nodes of partitioning of region V) whose elements are calculated by ensembling the magnitudes of the volumetric integrals

$$\int_{V^{(e)}} N_s N_i dV, \quad \int_{V^{(e)}} \operatorname{grad} N_s \operatorname{grad} N_i dV \quad (e = \overline{1, M}; \ s, i = \overline{1, K}^{(e)}); \tag{9}$$

$\{\dot{C}\}$ and $\{\Lambda\}$ are vectors of the nodal values of the quantities $\dot{C} = \partial C/\partial T$ and Λ; $\{S\}$ is the corresponding vector of the right-hand side of the equation. Eliminating vectors $\{\dot{C}\}$ and $\{\Lambda\}$ from system (8) and using the relations [6, 9]

$$\dot{C}_k = c(T_k)\rho(T_k)\dot{T}_k, \quad \Lambda_k = \int_{T_0}^{T_k} \lambda(x)\,dx = \kappa(T_k)\,T_k, \quad k = \overline{1, N}, \tag{10}$$

we come to a system of equations for the vector $\{T\}$ of the nodal values of the temperature:

$$[P][c]\{\dot{T}\} + ([B][\kappa] + [H])\{T\} = \{S\}, \tag{11}$$

where $[c]$ and $[\kappa]$ are diagonal matrices of the nodal values for the quantities $c(T)\rho(T)$ and $\kappa(T)$ (the question of the existence of the function κ in an analytical form that satisfies the last equality in system (10) is usually answered positively in practice, since the thermal conductivity coefficient λ of many materials can be approximated, for example, by a finite system of polynomials for the temperature).

Using approximation (2) for the temperature on the time interval $[t_n, t_{n+1}]$, we obtain from system (11) a system of algebraic equations similar to Eq. (3):

$$[A_*]\{T\}^{(n+1)} = \{R_*\}, \tag{12}$$

where

$$\begin{aligned} [A_*] &= [P][c]/\Delta t_{n+1} + \omega([B][\kappa] + [H]); \\ \{R_*\} &= ([P][c]/\Delta t_{n+1} - (1-\omega)([B][\kappa] + [H]))\{T\}^{(n)} + \{S\}. \end{aligned} \tag{13}$$

If for system (12) we use the same scheme as for system (3), then, as for Eq. (6), we write

$$[A_*]_i \{T\}_{i+1}^{(n+1)} = \{R_*\}_i \quad (i = 0, 1, 2, \ldots). \tag{14}$$

The matrix $[A_*]_i$ has a band structure, but it is not symmetric. Therefore, less economic methods (in the sense of computer time expenditures) are suitable for numerical solution of system of linear algebraic equations (14) than for system (6). However, as we can see from expressions (13) and (9), in order to form the matrix $[A_*]_i$, it is sufficient to calculate volumetric integrals (9) only once (before the beginning of the iterative process), and in each iteration it is necessary to find only the values of the functions $c(T)$, $\rho(T)$, and $\kappa(T)$ at the nodal points of the finite-element grid. Consequently, to form the matrix $[A_*]_i$, a smaller number of arithmetic operations is required than for forming the matrix $[A]_i$.

We will evaluate the computer time expenditures in forming and solving systems of linear algebraic equations (6) and (14). The process of the formation of these systems of equations can be divided into three stages: 1) calculation of volumetric integrals (4) and (9); 2) finding integrals (5); and 3) determination of the matrix elements of the system of linear algebraic equations and of the components of the vector in its right-hand side in terms of the calculated values of the volumetric and surface integrals, as well as the implementation of first-kind boundary conditions. We denote the computer time expenditures at each of the stages of the formation of systems

(6) and (14) by $\tau_1(A)$, $\tau_2(A)$, $\tau_3(A)$, and $\tau_1(A_*)$, $\tau_2(A_*)$, $\tau_3(A_*)$, respectively. Then for the values of computer time expenditures in forming systems of linear algebraic equations (6) and (14) for N_{it} iterations, $\tau_{form}(A)$ and $\tau_{form}(A_*)$, we write the following expressions:

$$\tau_{form}(A) = N_{it}[\tau_1(A) + \tau_2(A) + \tau_3(A)],$$
$$\tau_{form}(A_*) = \tau_1(A_*) + N_{it}[\tau_2(A_*) + \tau_3(A_*)]. \tag{15}$$

In order that the computer time expenditures in solving the heat-conduction problem by using scheme (11), i.e., $\tau(A_*)$, be smaller than expenditures with the traditional scheme of the finite-element method, i.e., $\tau(A)$, the following condition must be satisfied

$$\tau_{sol}(A_*) - \tau_{sol}(A) < \tau_{form}(A) - \tau_{form}(A_*). \tag{16}$$

Taking into account that $\tau_1(A) \approx \tau_1(A_*)$, $\tau_2(A) = \tau_2(A_*)$, $\tau_3(A_*) \approx \tau_3(A) << \tau_1(A)$ and using relations (15), we obtain from Eq. (16)

$$\tau_{sol}(A_*) - \tau_{sol}(A) < (N_{it} - 1)\tau_1(A). \tag{17}$$

We assume that for forming and solving systems of linear algebraic equations (6) and (14), effective algorithms are used in which the properties of the matrices are taken into account. The number of arithmetic operations carried out in solving a system of linear algebraic equations with a matrix of band structure by direct methods is proportional to $N\beta^2$ (β is the half-width of the band of the matrix) [5]. In addition, in solving a system of linear algebraic equations with an asymmetric matrix almost twice as many operations are performed than in solving a system with a symmetric matrix. Therefore, we can write

$$\tau_{sol}(A_*) - \tau_{sol}(A) = N_{it}\gamma_1 N\beta^2 \quad (\gamma_1 = \text{const} > 0). \tag{18}$$

The quantity $\tau_1(A)$ is proportional to M and to the number of computer operations $K^{(e)}_{oper}$ carried out in calculations of volumetric integrals (4), i.e.,

$$\tau_1(A) = \gamma_2 \sum_{e=1}^{M} K^{(e)}_{oper} \quad (\gamma_2 = \text{const} > 0). \tag{19}$$

The constants γ_1 and γ_2 are independent of N and β and have the dimension of time per computer operation.

As a result of the substitution of expressions (18) and (19) into Eq. (17) with allowance for the relation $N_{it} >> 1$, we obtain the following limitation on the magnitude of the half-width of the matrix band β:

$$\beta < \beta_{cr} \approx \left[(\gamma_2/(\gamma_1 N)) \sum_{e=1}^{M} K^{(e)}_{oper}\right]^{1/2}. \tag{20}$$

From this it follows that since N and M are quantities of the same order of magnitude, the quantity β_{cr} is mainly affected only by the number of computer operations performed in calculations of volumetric integrals (4). Thus, if the half-width of the band β in the matrix of the system of linear algebraic equations is smaller than a certain critical value β_{cr}, the use of scheme (11) for solving the nonlinear nonstationary heat-conduction problem requires less computer time than the standard scheme of the finite-element method.

To check this conclusion for an axially symmetric case, we carried out the following numerical experiment. When solving the heat-conduction problem using scheme (11) and the standard scheme of the finite-element method, we determined the ratio of the computer times $\tau(A_*)/\tau(A)$ for different values of the half-width of the band β in the matrix of the system of linear algebraic equations:

$$c(T)\frac{\partial T}{\partial t} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\lambda(T)\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\lambda(T)\frac{\partial T}{\partial z}\right)\right] + f(T, r, z, t)$$

$$(0 \le r, z \le 1,\ 0 < t \le 5.3),\ c(T) = 1 + c_1 T,\ \lambda(T) = 1 + \lambda_1 T,$$

$$T(r, z, 0) = 0,\ \frac{\partial T}{\partial r}(0, z, t) = \frac{\partial T}{\partial z}(r, 0, t) = 0,$$

$$\lambda\frac{\partial T}{\partial z}(r, 1, t) = q(r, t),\ \lambda\frac{\partial T}{\partial r}(1, z, t) = -\alpha(z, t)T(1, z, t),$$

$$f(T, r, z, t) = c(T) f_1(r, z, t) - \lambda(T) f_2(r, z, t) - f_3(r, z, t),$$

$$f_1(r, z, t) = ar^2(1.1 - r)(1 + pz^2)/(1 + t)^2 + b, \tag{21}$$

$$f_2(r, z, t) = at[(4.4 - 9r)(1 + pz^2) + 2pr^2(1.1 - r)]/(1 + t),$$

$$f_3(r, z, t) = \lambda_1(art/(1 + t))^2[(2.2 - 3r)^2(1 + pz^2)^2 +$$

$$+ (2prz(1.1 - r))^2],$$

$$q(r, t) = 2pF(r, t)[1 + \lambda_1((1 + p)F(r, t) + bt],$$

$$\alpha(z, t) = 8(1 - b/H(z, t))(1 + \lambda_1 tH(z, t)),$$

$$F(r, t) = ar^2(1.1 - r)t/(1 + t),\ H(z, t) = 0.1a(1 + pz^2)/(1 + t) + b$$

$$(c_1 = 0.3,\ \lambda_1 = -0.1,\ a = 10,\ p = 3,\ b = 0.2)$$

We call the quantity $\xi = \tau(A_*)/\tau(A)$ the relative computer time expenditure. Both schemes were implemented in FORTRAN, and the calculations were performed on an AT/386 PC. The space rdomain was divided uniformly into $M = m \times m$ finite elements, and the time domain, into 30 intervals $[t_n, t_{n+1}]$ $(n = 0, 1, ..., 29)$ with a variable length Δt_n that varied from $\Delta t_1 = 0.05$ to $\Delta t_{30} = 0.3$; $\omega = 2/3$. For digitization in time, a two-step procedure [6] was used over each interval $[t_n, t_{n+1}]$. The width of the matrix band in the system of linear algebraic equations was minimized using a subroutine for minimization of the width of the band for rectangular regions (these cases are marked by an asterisk in Table 1) and the algorithm of [10]. The volumetric and surface integrals were calculated by the Gauss quadrature formulas with three points of integration for each variable. We used the modified Kraut method [4] to solve system of linear algebraic equations (6) and the Gauss method with selection of the main element in the matrix columns [11] to solve system of linear algebraic equations (14). The numerical solutions of the problem obtained using each of the indicated schemes of the finite-element method were compared with the exact solution

$$T(r, z, t) = ar^2(1.1 - r)(1 + pz^2)t/(1 + t) + bt.$$

Table 1 presents the relative computer times ξ for different variants of partitioning of the region $0 \le r, z \le 1$ into finite elements and different values of β for the half-width of the matrix band in the system of linear algebraic equations. It is evident that ξ increases in proportional to β, reaching 1.0 at $\beta = \beta_{cr} \approx 47$. This means that when $\beta < 47$, the use of scheme (11) for solving axially symmetric nonlinear nonstationary heat-conduction problems is more advantageous in the sense of computer time expenditures as compared to the standard scheme of the finite-element method. We note that in solving a number of practical problems, the half-width of the matrix band in the system of linear algebraic equations turns out to be smaller than β_{cr}. So, in calculation of axially

TABLE 1. Relative Computer Time Expenditures ξ for Different Values of the Half-Width β of the Matrix Band in Resolving System of Linear Algebraic Equations

M	β	ξ	M	β	ξ
144*	15	0.50	484	39	0.87
144	23	0.57	576	42	0.94
400*	23	0.58	676	45	0.97
324	32	0.73	841	50	1.03
400	35	0.80			

TABLE 2. Relative Error δ in Numerical Solutions of Problem (21) Obtained Using Scheme (11) and the Standard Scheme of the Finite-Element Method

Time, t	δ, %		Time, t	δ, %	
	scheme (11)	standard scheme of FEM		scheme (11)	standard scheme of FEM
0.05	13.10	13.10	1.75	1.38	1.50
0.25	5.12	5.14	2.55	1.14	1.28
0.40	3.14	3.18	3.55	1.09	1.26
0.60	2.49	2.55	4.70	1.06	1.25
0.80	1.86	1.93	5.30	1.03	1.24
1.40	1.44	1.55			

symmetric thermal fields in an installation for hot molding of powdered ceramic materials [9], the half-width of the matrix band was equal to 20, i.e., in the axially symmetric case the value β_{cr} is sufficient for ensuring the efficiency of scheme (11) when solving a number of practical problems of heat-conduction theory.

Along with establishment of the efficiency of scheme (11), it is also important to attain good accuracy of the numerical solution obtained. The values of the relative error δ in the numerical solutions of problem (21) obtained by using scheme (11) and the standard scheme of the finite-element method are given in Table 2 for the case of $M = 144$ and $\beta = 15$. The error δ was calculated in the norm of the space $L_2(V)$ [7]. It is seen that even on a rather coarse finite-element grid and with rather large time steps, scheme (11) has good accuracy compared to the standard scheme of the finite-element method.

In conclusion we note that the considered example of heat-conduction problem (21) includes the temperature dependence of the thermophysical properties of the material, volumetric and surface heat sources, and heat transfer on the body surface, i.e., conditions that are often encoutered in practice in solving heat-conduction problems. Taking into account this fact and using the data of Tables 1 and 2, we may conclude that in an axially symmetric case the variant of the finite-element method suggested in [6] can be effective (in the sense of computer time expenditures) for solving a wide range of practical problems of heat-conduction theory.

NOTATION

r, radius-vector of a point of spatial region V; t, time; c, ρ, λ, heat capacity, density, thermal conductivity coefficient; α, convective heat-transfer coefficient; $T_{in}(r)$, initial temperature; Σ_1, Σ_2, Σ_3, nonintersecting parts of the external surface of the region V; $\varphi(r, t)$, given temperature distribution on the surface Σ_1; $T_{med}(r, t)$, temperature of the external medium; f, q, intensities of the volumetric and surface heat sources, respectively; N_s, form function; r, z, cylindrical coordinates.

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